

Quantum dynamical entropy, chaotic unitaries and complex Hadamard matrices

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Abstract—We introduce two information-theoretical invariants for the projective unitary group acting on a finite-dimensional complex Hilbert space: PVM- and POVM-dynamical (quantum) entropies. They quantify the randomness of the successive quantum measurement results in the case where the evolution of the system between each two consecutive measurements is described by a given unitary operator. We study the class of chaotic unitaries, i.e., the ones of maximal entropy or, equivalently, such that they can be represented by suitably rescaled complex Hadamard matrices in some orthonormal bases. We provide necessary conditions for a unitary operator to be chaotic, which become also sufficient for qubits and qutrits. These conditions are expressed in terms of the relation between the trace and the determinant of the operator. We also compute the volume of the set of chaotic unitaries in dimensions two and three, and the average PVM-dynamical entropy over the unitary group in dimension two. We prove that this mean value behaves as the logarithm of the dimension of the Hilbert space, which implies that the probability that the dynamical entropy of a unitary is almost as large as possible approaches unity as the dimension tends to infinity.

Index Terms—quantum dynamical entropy, quantum measurement, projection valued measure (PVM), positive operator valued measure (POVM), unitary operator, complex Hadamard matrix.

I. INTRODUCTION

Imagine we are standing somewhere on the Earth's (spherical) surface that rotates around the north-south axis. Try to choose this place in such a way as to make as large as possible the angle between the axis passing through the chosen point and the centre of the Earth, and the rotated axis determined after some fixed time interval. If the time period is less than six hours, the choice is simple: we must locate ourselves somewhere on the equator. However, if the elapsed time is chosen between six and twelve hours, the situation becomes more complicated. We have to travel north (or south) the equator, eventually reaching, for twelve hours, the 45th parallel north (say on the border between Montana and Wyoming) or the 45th parallel south (e.g., in Becks, a small settlement on the South Island of New Zealand). In the former 'short time' case, the maximal attainable angle is equal to the earth's angle of rotation, but in the latter, i.e., when the time is long enough, we can always find a point on the Earth (or, more precisely, a circle of latitude) such that the angle between the two lines in question is right. Now, if we swap the Earth for the Bloch sphere representing qubits, this simple riddle illustrates the difference between two kinds of unitary transformations (represented here as Bloch sphere rotations): non-chaotic and chaotic ones. The exploration of this difference is the main theme of this paper.

The invariant that we shall use to distinguish chaotic unitaries is quantum dynamical entropy. The notion of (classical) *dynamical entropy* (or *entropy rate*) is due to Claude E. Shannon [35], [36], who introduced it into information theory, and Andrei Kolmogorov [27], who made it a basic tool for studying dynamical systems. In his seminal paper Shannon discussed the problem of computing entropy for a discrete and ergodic information source sending messages to a receiver. This quantity can be determined from the statistics of finite message sequences, namely, it is the limit of entropy of a block of symbols divided by its length, or the limit of conditional entropy of the next symbol given the preceding ones, as the block length tends to infinity. In the Kolmogorov-Sinai theory the definition of entropy is

very similar to Shannon's, except that instead of message sequences, the results of discrete measurements (represented there by finite partitions of the phase space) are analysed and then the supremum over all such measurements is taken. The entropy defined in this way is invariant with respect to metric isomorphisms of dynamical systems. Moreover, using the Kolmogorov-Sinai (KS) entropy, we can formally distinguish *regular* systems (with dynamical entropy equal to zero) from *chaotic systems* (with strictly positive dynamical entropy).

In the present paper we consider a quantum analogue of this notion. We analyse the situation where successive measurements are performed on a finite-dimensional quantum mechanical system whose evolution between two subsequent measurements is given by a quantum operation. We assume that the dynamics of the quantum system is described by a finite-dimensional unitary operator, and the measurement process either by a *von Neumann-Lüders instrument* (represented by a projection valued measure - PVM) or by a *generalised Lüders instruments*, disturbing the initial state in the minimal way (represented by a positive operator valued measure - POVM). If the measure consists of rank-1 operators, then such process generates two Markov chains: the first one in the space of states (so-called *discrete quantum trajectories*, see, e.g., [30], [29], [4]), and the second one in the space of measurement outcomes. The dynamical entropy (entropy rate) of the latter can be used to estimate the randomness of the measurement results.

Understood in this way, quantum dynamical entropy was introduced independently by Srinivas [41], Pechukas [34], Beck and Graudenz [6], see also [1], and recently rediscovered and analysed by Crutchfield and Wiesner under the name of *quantum entropy rate* [16], [47]. It is also closely related to the *entropy of unitary matrices*, used in different contexts by various authors [26], [50], [2]. Imitating the definition of the Kolmogorov-Sinai entropy and taking the supremum over the class of PVM measurements, we get the *PVM-dynamical entropy*, which depends only on the quantum dynamics and characterizes its ability to produce random sequences of measurement outcomes. In the case of POVMs the situation is more complicated as there are two independent sources of randomness that can influence the value of dynamical entropy. The first is the underlying dynamics of the system, described by a unitary operator. The second is the POVM measurement, which potentially introduces some additional randomness. Subtracting the dynamical entropy calculated for trivial (identity) dynamics from the original entropy rate, and then taking the supremum over the class of POVM measurements, we get another quantity, the *POVM-dynamical entropy*, which again depends only on the unitary operator and is larger than or equal to its PVM counterpart. These measurement independent definitions of dynamical quantum entropy for finite-dimensional systems were introduced in a more general setting in [39], [28], [40] and then developed further in [37], [38]. However, only preliminary results have been obtained so far. In the present paper we study the notion of PVM-dynamical entropy in full details, postponing more comprehensive analysis of the POVM-dynamical entropy to further publications.

Note that a widely accepted generalization of KS entropy for quantum mechanics has not yet been found, in spite of the fact that several attempts to define such a quantity have been made [33], [11]. In particular, the best-known quantum dynamical entropies, such as the Connes-Narnhofer-Thirring (CNT) entropy [13] or the Alicki-Fannes (AF) [3] entropy, vanish for finite-dimensional quantum systems [8], [7, Sec. 14.5], and so they cannot be used to quantify the randomness of the successive measurement outcomes in the case we study here.

In Sec. II we introduce the notions of PVM- and POVM-dynamical entropy and observe that they are invariant under conjugation, inversion and phase multiplication, which makes them class functions for the projective unitary-antiunitary group. These quantities are non-negative and bounded from above by the logarithm of the dimension of the underlying Hilbert space, also called the number of degrees of freedom of the quantum mechanical system. We show that their mean values averaged over the unitary ensemble are only slightly smaller than the upper bound and so tend logarithmically to infinity. In Sec. III we use these dynamical entropies to distinguish between *chaotic*, i.e., the ones of maximal entropy, and non-chaotic unitaries. The former are characterized as those that can be represented by a suitably rescaled complex Hadamard matrix in some orthonormal basis. In Sec. IV we compute the volume of the set of chaotic matrices as well as the exact value of mean PVM-dynamical entropy in dimension two. Sec. V contains a necessary condition for a unitary matrix to be chaotic. We show that for qubits and qutrits this condition is, in fact, sufficient. This fact allows us to compute the volume of the set of chaotic matrices also in dimension three. In Sec. VI we discuss the difficulties that arise when trying to extend the definition of quantum dynamical entropy to the realm of general measurements.

II. QUANTUM DYNAMICAL ENTROPY - DEFINITION AND BASIC PROPERTIES

We assume that the pure states of a d -dimensional quantum system are represented by the complex projective space \mathbb{CP}^{d-1} or, equivalently, by the set $\mathcal{P}(\mathbb{C}^d)$ of one-dimensional projections in \mathbb{C}^d . The set of mixed states $\mathcal{S}(\mathbb{C}^d)$ is the convex closure of $\mathcal{P}(\mathbb{C}^d)$, i.e., the set of density (Hermitian, positive semi-definite, and trace one) operators on \mathbb{C}^d .

The *measurement* (with k possible outcomes) of this system is given by a *positive operator valued measure (POVM)*, i.e., an ensemble of positive (non-zero) Hermitian operators Π_j ($j = 1, \dots, k$) on \mathbb{C}^d that sum to the identity operator, i.e., $\sum_{j=1}^k \Pi_j = \mathbb{I}$. For simplicity we shall consider here only *normalized rank-1 POVMs*, where Π_j ($j = 1, \dots, k$) are rank-1 operators and $\text{tr}(\Pi_j) = \text{const}(j) = d/k$. Necessarily, $k \geq d$ in this case, and there exists an ensemble of pure states $|\varphi_j\rangle \langle \varphi_j| \in \mathcal{P}(\mathbb{C}^d)$ ($j = 1, \dots, k$) such that $\Pi_j = (d/k) |\varphi_j\rangle \langle \varphi_j|$. Thus, $\sum_{j=1}^k |\varphi_j\rangle \langle \varphi_j| = (k/d) \cdot \mathbb{I}$. In particular, if $k = d$ and so $(\varphi_j)_{j=1}^d$ is an orthonormal basis of \mathbb{C}^d , we get a special class of *projection valued measures (PVMs)*.

If the state of the system before the measurement (the *input state*) is $\rho \in \mathcal{S}(\mathbb{C}^d)$, then the probability $p_j(\Pi, \rho)$ of the j -th outcome is given by $p_j(\Pi, \rho) := \text{tr}(\rho \Pi_j)$ for $j = 1, \dots, k$. In particular, for normalized rank-1 POVMs we have $p_j(\Pi, \rho) = (d/k) \langle \varphi_j | \rho | \varphi_j \rangle$, and if $\rho = |\psi\rangle \langle \psi| \in \mathcal{P}(\mathbb{C}^d)$, then $p_j(\Pi, \rho) = (d/k) |\langle \varphi_j | \psi \rangle|^2$ (the *Born rule*). The measurement process generically alters the state of the system, but the POVM alone is not sufficient to determine the post-measurement (or *output*) state. This can be done by defining a *measurement instrument* (in the sense of Davies and Lewis [19]) compatible with Π , see also [22, Ch. 5]. We shall only consider here the so-called *generalised Lüders instruments*, disturbing the initial state in the minimal way, see [20, p.404], where the output state is $|\varphi_j\rangle \langle \varphi_j|$, providing the result of the measurement was j .

The successive measurements described by Π are performed on an evolving quantum system. We assume that the motion of the system between two subsequent measurements is governed by $U \in \mathcal{U}(d)$ acting as $\mathcal{S}(\mathbb{C}^d) \ni \rho \rightarrow U\rho U^* \in \mathcal{S}(\mathbb{C}^d)$. Then the results of consecutive measurements are represented by finite strings of letters from a k -element alphabet. The probability of obtaining the string (i_1, \dots, i_n) , where $i_m = 1, \dots, k$ for $m = 1, \dots, n$, and $n \in \mathbb{N}$, is then given by the Wigner formula [48]

$$P_{i_1, \dots, i_n}(\rho) := p_{i_1}(\rho) \cdot \prod_{m=1}^{n-1} p_{i_{m+1}}(\rho_{i_m}),$$

where ρ is the initial state of the system, $p_j(\rho) := (d/k) \langle \varphi_j | \rho | \varphi_j \rangle$ is the probability of obtaining j in the first measurement, and $p_{jl} := (d/k) |\langle \varphi_j | U | \varphi_l \rangle|^2$ is the probability of getting l as the result of the measurement, providing the result of the preceding measurement was j , for $j, l = 1, \dots, k$. In consequence, the combined evolution of states is Markovian with the initial distribution given by $p := (p_j)_{j=1}^k$ and the transition matrix $P := (p_{jl})_{j,l=1}^k$ [39], [37].

The randomness of the measurement outcomes can be analysed with the help of *quantum entropy of U with respect to Π* defined in an analogous way to the Kolmogorov-Sinai entropy, namely

$$H(U, \Pi) := \lim_{n \rightarrow \infty} (H_{n+1} - H_n) = \lim_{n \rightarrow \infty} \frac{H_n}{n},$$

where

$$H_n := \sum_{j_1, \dots, j_n=1}^k \eta(P_{j_1, \dots, j_n}(\rho_*))$$

with the *Shannon function* $\eta: \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $\eta(x) := -x \ln x$ for $x > 0$ and $\eta(0) := 0$. The maximally mixed state $\rho_* = \mathbb{I}/d$ plays here the role of the ‘stationary state’ for Markov evolution with $p_j(\rho_*) = 1/k$ for $j = 1, \dots, k$ [39], [37].

Using the formula for the entropy of a Markov chain, which is a special case of a much more general *integral entropy formula* [37], it is easy to show [38, eq. (24)] that

$$\begin{aligned} H(U, \Pi) &= \frac{1}{k} \sum_{j,l=1}^k \eta(p_{jl}) \\ &= \ln \frac{k}{d} + \frac{d}{k^2} \sum_{j,l=1}^k \eta(|\langle \varphi_j | U | \varphi_l \rangle|^2). \end{aligned} \quad (1)$$

In consequence,

$$\ln(k/d) \leq H(U, \Pi) \leq \ln k. \quad (2)$$

There are two sources of randomness in this model, the measurement process and the underlying unitary dynamics, and we would like to quantify their impact separately. This can be done by defining two quantities:

- the *measurement entropy* of Π given by

$$H_{\text{meas}}(\Pi) := H(\mathbb{I}, \Pi);$$

- the *dynamical entropy of U with respect to Π* given by

$$H_{\text{dyn}}(U, \Pi) := H(U, \Pi) - H_{\text{meas}}(\Pi).$$

Now, we introduce two kinds of measurement independent quantum dynamical entropies, by maximizing H_{dyn} either over all PVMs or all POVMs. Namely, the *PVM-dynamical entropy of U* :

$$H_{\text{dyn}}(U) := \max_{\Pi \in \text{PVM}} H_{\text{dyn}}(U, \Pi), \quad (3)$$

and the *POVM-dynamical entropy of U* :

$$\overline{H}_{\text{dyn}}(U) := \sup_{\Pi \in \text{POVM}} H_{\text{dyn}}(U, \Pi). \quad (4)$$

Analogously, we can define quantum dynamical entropy for an antiunitary transformation.

Note that the supremum is attainable in (3), since the set of all PVMs, i.e., all projective orthonormal (ordered) bases, forms a compact space isomorphic to the $d(d-1)$ -dimensional flag manifold $U(d)/U(1)^d$ [9, p.133] and it follows from (1) that $H_{\text{dyn}}(U, \cdot)$ is continuous in this space. For every $U \in U(d)$ we have $\min_{\Pi \in \text{PVM}} H(U, \Pi) = 0$, since the PVM Π generated with the help of an eigenbasis of U gives $H(U, \Pi) = 0$. In consequence, we cannot define here a quantum counterpart of classical *Kolmogorov automorphisms* (*K-systems*), i.e., maps with positive entropy with respect to all non-trivial finite partitions of the phase space.

For $\Pi \in \text{PVM}$ we have $H_{\text{meas}}(\Pi) = 0$, and so $H_{\text{dyn}}(U, \Pi) = H(U, \Pi)$. Consequently, we get

$$H_{\text{dyn}}(U) = \max_{\Pi \in \text{PVM}} H(U, \Pi),$$

which implies

$$H_{\text{dyn}}(U) = \max_{(e_j)_{j=1}^d} \frac{1}{d} \sum_{j,l=1}^d \eta(|\langle e_j | U | e_l \rangle|^2),$$

where the maximum is taken over all orthonormal bases. Equivalently, we can fix a basis (e.g., an eigenbasis of U) and take the maximum over all unitary transformations:

$$H_{\text{dyn}}(U) = \max_{V \in U(d)} \frac{1}{d} \sum_{j,l=1}^d \eta(|(V^* U V)_{jl}|^2). \quad (5)$$

Moreover, from (2) we get $|H_{\text{dyn}}(U, \Pi)| \leq \ln d$ and

$$0 \leq H_{\text{dyn}}(U) \leq \overline{H}_{\text{dyn}}(U) \leq \ln d.$$

The bounds are achievable, as we have $\overline{H}_{\text{dyn}}(\mathbb{I}) = 0$ and $H_{\text{dyn}}(F_d/\sqrt{d}) = \ln d$, where F_d is a unitary operator represented in some basis by the *Fourier matrix* of size d , given by $(\omega_d^{(j-1)(l-1)})_{j,l=1}^d$ with $\omega_d := \exp(2\pi i/d)$.

The following proposition that summarizes facts concerning invariance of the dynamical entropies is easy to show.

Proposition 1 (invariance). *The dynamical entropies H_{dyn} and $\overline{H}_{\text{dyn}}$ are invariant under the following operations*

- (i) *conjugation*: $U \rightarrow V^{-1}UV$ for every unitary or antiunitary V ;
- (ii) *inversion*: $U \rightarrow U^{-1}$;
- (iii) *phase multiplication*: $U \rightarrow e^{i\varphi}U$ for $\varphi \in \mathbb{R}$.

It follows from condition (i) above that both these quantities are unitary (and antiunitary) invariants (i.e., unitary class functions), and so they depend only on the spectrum of U , since two unitary matrices are unitarily similar if and only if they have the same spectrum (treated as a multiset). The space of conjugacy classes of unitary matrices is isomorphic to $SP^d(S^1)$, i.e., the d -th symmetric product of S^1 , which is a fibre bundle over S^1 and the fibres are $(d-1)$ -dimensional simplices [32]. According to (iii), both quantum dynamical entropies are also projective invariants, and so they can be treated as class functions for the projective unitary-antiunitary group.

To lower bound the mean value of the PVM-dynamical entropy averaged over the ensemble of unitary matrices, we consider yet another unitary invariant, the *PVM-average dynamical entropy*, given by $M(U) := \langle H(U, \Pi) \rangle_{\Pi \in \text{PVM}}$ for $U \in U(d)$. Namely, we have

Theorem 2 (mean entropy bounds).

$$\ln d - (1 - \gamma) < \sum_{k=2}^d \frac{1}{k} = \langle M(U) \rangle_{U(d)} < \langle H_{\text{dyn}}(U) \rangle_{U(d)} < \ln d,$$

where $\gamma \approx 0.577$ is Euler's constant.

Proof: All entropies are bounded from above by $\ln d$. On the other hand, from Jones ([23, eq.(13)] and [24, eq.(27)]), see also [40], [50], we deduce that $\langle H(U, \Pi) \rangle_{U(d)} = \sum_{k=2}^d \frac{1}{k}$ for every $\Pi \in \text{PVM}$. Hence

$$\begin{aligned} \gamma - 1 + \ln d &< \sum_{k=2}^d \frac{1}{k} = \langle M(U) \rangle_{U(d)} \\ &= \max_{\Pi \in \text{PVM}} \langle H(U, \Pi) \rangle_{U(d)} \leq \left\langle \max_{\Pi \in \text{PVM}} H(U, \Pi) \right\rangle_{U(d)} \\ &= \langle H_{\text{dyn}}(U) \rangle_{U(d)} \leq \ln d. \end{aligned}$$

From the continuity of H_{dyn} it follows that the last two inequalities are strict. ■

In consequence, we see that the mean values of both entropies H_{dyn} and $\overline{H}_{\text{dyn}}$ are almost as large as possible and increase logarithmically with the dimension of the Hilbert space. Moreover, from Chebyshev's inequality we deduce that the probability of $H_{\text{dyn}} \geq \ln d - f(d)$ tends to 1, providing $f(d) \rightarrow \infty$, even if the latter convergence is very slow. In Sect. IV we compute the exact value of $\langle H_{\text{dyn}}(U) \rangle_{U(d)}$ for $d = 2$.

III. ENTROPY-MAXIMISING UNITARIES

The concept of quantum dynamical entropy specifies a special class of entropy-maximising unitaries, such as the Fourier unitaries mentioned above. We shall call them *chaotic* since they can be used to produce maximally random sequences of measurement results. As we shall see, this property does not depend on which of the two definitions we work with. Namely, from (2) it follows that

$$H_{\text{dyn}}(U, \Pi) = \ln d \text{ iff } H(U, \Pi) = \ln k \text{ and } H(\mathbb{I}, \Pi) = \ln(k/d). \quad (6)$$

By (1), we get $H(\mathbb{I}, \Pi) = \ln(k/d)$ if and only if Π is a PVM, i.e., $k = d$, and then, clearly, $H(\mathbb{I}, \Pi) = 0$. Thus,

$$\overline{H}_{\text{dyn}}(U) = \ln d \text{ iff } H_{\text{dyn}}(U) = \ln d.$$

Moreover, chaotic unitaries turn out to be exactly those that are represented by a suitably rescaled complex *Hadamard matrix* in some basis.

Proposition 3. *Let $U \in U(d)$. Then the following conditions are equivalent:*

- (i) *U is chaotic;*
- (ii) *there exists an orthonormal basis $\{e_j\}_{j=1}^d$ such that $\frac{1}{d} \sum_{j,l=1}^d \eta(|\langle e_j | U | e_l \rangle|^2) = \ln d$;*
- (iii) *there exists an orthonormal basis $\{e_j\}_{j=1}^d$ such that $\{e_j\}_{j=1}^d$ and $\{Ue_j\}_{j=1}^d$ are mutually unbiased;*
- (iv) *there exists an orthonormal basis $\{e_j\}_{j=1}^d$ such that $\sum_{j,l=1}^d |\langle e_j | U | e_l \rangle| = d\sqrt{d}$;*
- (v) *$\sqrt{d}U$ is represented by a complex Hadamard matrix in some orthonormal basis $\{e_j\}_{j=1}^d$, i.e., $|\langle e_j | U | e_l \rangle| = 1/\sqrt{d}$ for each $j, l = 1, \dots, d$.*

Proof: The equivalence of (i) and (ii) follows immediately from (2) and (6). As the Shannon entropy is maximal only for the uniform probability, all expressions of the form $|\langle e_j | U | e_l \rangle|^2$ for $j, l = 1, \dots, d$ must be equal, which proves the equivalence of (ii) and (v). On the other hand, (v) is just (iii) expressed in another way. The equivalence of (iv) and (v) follows from [5]. ■

The fact that Hadamard matrices saturate the upper bound for the so-called entropy of a unitary matrix is well known [50]. Observe, however, that the analogous problem for real orthogonal matrices is highly non-trivial [21], since real Hadamard matrices can exist only if $d = 1, 2$ or is a multiple of 4.

From Proposition 3 we deduce immediately a simple necessary condition for U to be chaotic.

Corollary 4. *If $U \in \mathcal{U}(d)$ is chaotic, then $|\text{tr } U| \leq \sqrt{d}$.*

We shall see in Sec. V that for $d = 2$, in contrast to higher dimensions, this condition is also sufficient.

As *quantum gates* are represented by unitaries (defined up to a phase) we can talk about *dynamical entropies of quantum gates* and we can distinguish the class of *chaotic quantum gates*. Using the formula for dynamical entropy presented in the next section, we shall show that among chaotic unitaries one can list many well-known quantum gates, including the Hadamard, NOT (Pauli- X), Pauli- Y , Phase Flip (Pauli- Z), $\pi/4$ -phase shift and $\sqrt{\text{NOT}}$ gates in dimension two. Also the CNOT (XOR) and SWAP gates in dimension four belong to this class. To see this, observe that both these matrices are unitarily equivalent to $U := \text{diag}(1, 1, 1, -1)$. The spectra of U and the real-valued Hadamard matrix $F_4^{(1)}(3\pi/2)$, see [45], coincide. Thus, by Proposition 3, U is chaotic. On the other hand, the $\pi/8$ -phase shift gate in dimension two as well as the $\sqrt{\text{CNOT}}$ and $\sqrt{\text{SWAP}}$ gates in dimension four do not fulfill the trace condition from Corollary 4, and so they are not chaotic. It follows also from this corollary that the CNOT gate is the only chaotic gate among controlled- U gates in dimension four. In the same way we argue that multiqubit controlled gates, like Toffoli (CCNOT), Fredkin (CSWAP) or Deutsch (CCR) gates in dimension eight, cannot be chaotic.

IV. PVM-DYNAMICAL ENTROPY: QUBITS

Computing the PVM-dynamical entropy in dimension two is a relatively easy task, as the optimization problem reduces to finding the maxima of real-valued functions belonging to a one-parameter family. Here, the parameter is the angle between two eigenvalues of a unitary map. The formula for entropy in this case has been already obtained in [37], but for the sake of completeness we recall hereafter its proof.

Let $U \in \mathcal{U}(\mathbb{C}^2)$ with the spectrum $\{\exp(i\varphi), \exp(i\psi)\}$, where $\varphi, \psi \in [0, 2\pi)$. Fix an eigenbasis of U . In this basis U is represented by the matrix

$$U \sim \begin{bmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(i\psi) \end{bmatrix}.$$

Consider now $V \in \mathcal{U}(\mathbb{C}^2)$ given by

$$V \sim \begin{bmatrix} u & v \\ w & z \end{bmatrix},$$

where $u, v, w, z \in \mathbb{C}$ satisfy $|u|^2 + |v|^2 = |w|^2 + |z|^2 = 1$ and $u\bar{w} + v\bar{z} = 0$. Then

$$V^*UV \sim \begin{bmatrix} |u|^2 e^{i\varphi} + |w|^2 e^{i\psi} & v\bar{u}e^{i\varphi} + z\bar{w}e^{i\psi} \\ v\bar{u}e^{i\varphi} + z\bar{w}e^{i\psi} & |v|^2 e^{i\varphi} + |z|^2 e^{i\psi} \end{bmatrix}.$$

Put $p := |u|^2 \in [0, 1]$, $\theta := |\varphi - \psi| \bmod \pi$, and $c := \sin^2(\theta/2) \in [0, 1]$. As $|z|^2 = p$ and $|w|^2 = |v|^2 = 1 - p$, we obtain

$$\frac{1}{2} \sum_{j,l=1}^2 \eta(|(V^*UV)_{jl}|^2) = \eta(4p(1-p)c) + \eta(1-4p(1-p)c). \quad (7)$$

Denote the right-hand side of (7) by $h_c(p)$. Then $h_c: [0, 1] \rightarrow \mathbb{R}$ attains the maximum equal to $\ln 2$ at $\frac{1}{2}(1 \pm \sqrt{1 - (2c)^{-1}})$ for $c \geq 1/2$, and equal to $\eta(c) + \eta(1-c)$ at $1/2$ for $c \leq 1/2$. Using this fact and (5), we obtain

Proposition 5.

$$H_{\text{dyn}}(U) = \begin{cases} \ln 2 & \theta \geq \frac{\pi}{2} \\ \eta(\cos^2(\frac{\theta}{2})) + \eta(\sin^2(\frac{\theta}{2})) & \theta \leq \frac{\pi}{2} \end{cases}. \quad (8)$$

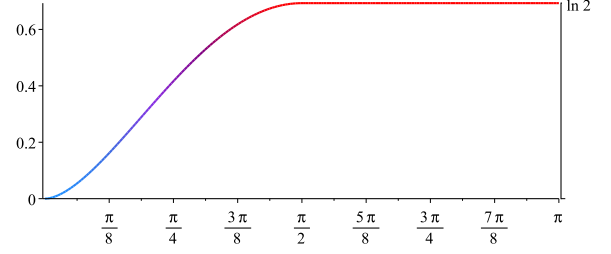


Fig. 1. The dependence of H_{dyn} on θ (the chaotic part in red).

Denote by $\{|0\rangle, |1\rangle\}$ the eigenbasis of U . Observe that

- the critical point at which H_{dyn} hits its maximum possible value $\ln 2$ is $\theta = \frac{\pi}{2}$; this applies to well-known $\pi/4$ -phase shift and $\sqrt{\text{NOT}}$ gates;
- the PVMs with respect to which $H(U, \Pi)$ attains its maximal value are given by the bases $\{|x^\tau\rangle, |x_\perp^\tau\rangle\}$ defined as $|x^\tau\rangle := \frac{1}{\sqrt{2}}(|0\rangle + e^{i\tau}|1\rangle)$ for $\theta \leq \frac{\pi}{2}$ and $|x^\tau\rangle := \sqrt{r}|0\rangle + e^{i\tau}\sqrt{1-r}|1\rangle$, with $r := \frac{1}{2}(1 \pm \sqrt{1 - (2\sin^2(\theta/2))^{-1}})$, for $\theta \geq \frac{\pi}{2}$, where τ is an arbitrary number from $[0, 2\pi)$.

The geometric interpretation of the latter fact, mentioned already in the introduction, is the following. Fix the Bloch vectors corresponding to the eigenbasis of U as the north and south poles of the Bloch sphere. Then U can be interpreted as the rotation around the north-south axis by the angle $\theta := \varphi - \psi$. Under this picture, finding a maximizing PVM is equivalent to choosing the appropriate axis such that the angle between this axis and its image under the rotation is maximal. If θ is acute, then the axis must lie in the equatorial plane and the angle in question is equal to θ , but if θ is obtuse, we can find an axis that can be transformed into a perpendicular thereto by the rotation.

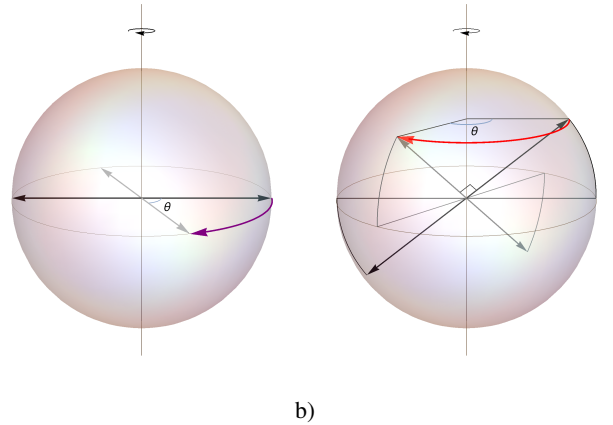


Fig. 2. Maximizers for the PVM-dynamical entropy in dimension $d = 2$, where the unitary map is represented in the Bloch sphere as a rotation by the angle: a) acute (purple) and b) obtuse (red).

Next, we compute the volume of the set of chaotic operators in the ensemble of unitary matrices as well as the average value of the PVM-dynamical entropy. To this aim we use the Weyl integration formula for $\mathcal{U}(d)$ group [46, Theorem 7.4.B]. Recall that $F: \mathcal{U}(d) \rightarrow \mathbb{C}$ is a *class function* if it is constant on the conjugacy classes, i.e., for all $U, V \in \mathcal{U}(d)$ we have $F(U) = F(V^*UV)$.

Theorem (Weyl's integration formula). *If $F \in L^1(\mathcal{U}(d))$ is a class function, then the following formula holds*

$$\begin{aligned} & \int_{\mathcal{U}(d)} F(U) dm(U) = \\ &= \frac{1}{d! (2\pi)^d} \int_{[0, 2\pi]^d} f(\theta_1, \dots, \theta_d) \prod_{1 \leq j < l \leq d} |e^{i\theta_j} - e^{i\theta_l}|^2 d\theta_1 \dots d\theta_d, \end{aligned}$$

where m denotes the normalized Haar measure on $\mathcal{U}(d)$ and $f(\theta_1, \dots, \theta_d) := F(\Theta)$ for $\Theta := \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_d})$.

Applying this formula, we get

Theorem 6. *Let $C_2 := \{U \in \mathcal{U}(2) : U \text{ is chaotic}\}$. Then*

$$m(C_2) = \frac{1}{2} + \frac{1}{\pi} \approx 0.8183.$$

Proof: It follows from the Weyl integration formula that

$$\begin{aligned} m(C_2) &= \int_{\mathcal{U}(2)} \mathbf{1}_{C_2}(U) dm(U) \\ &= \frac{1}{4\pi} \int_{\pi/2}^{3\pi/2} |e^{i\varphi} - 1|^2 d\varphi \\ &= \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} (1 - \cos \varphi) d\varphi = \frac{1}{2} + \frac{1}{\pi}, \end{aligned}$$

as desired. \blacksquare

Theorem 7. *The average value of the PVM-dynamical entropy is given by*

$$\langle H_{\text{dyn}}(U) \rangle_{\mathcal{U}(2)} = \frac{3}{2} \ln 2 - \frac{1}{2} - \frac{1}{2\pi} + \frac{C}{\pi} \approx 0.672,$$

where C is Catalan's constant, which may be computed from the formula

$$C := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.916.$$

Proof: Using again the Weyl integration formula and (8), we get

$$\begin{aligned} & \langle H_{\text{dyn}}(U) \rangle_{\mathcal{U}(2)} \\ &= \int_{\mathcal{U}(d)} H_{\text{dyn}}(U) dm(U) \\ &= \frac{1}{\pi} \int_0^{\pi/2} \left(\eta\left(\cos^2\left(\frac{\varphi}{2}\right)\right) + \eta\left(\sin^2\left(\frac{\varphi}{2}\right)\right) \right) (1 - \cos \varphi) d\varphi \\ &+ \left(\frac{1}{2} + \frac{1}{\pi} \right) \ln 2. \end{aligned}$$

The first summand can be written as the sum of several integrals, which gives

$$\begin{aligned} & \langle H_{\text{dyn}}(U) \rangle_{\mathcal{U}(2)} \\ &= \ln 2 + \frac{1}{\pi} \int_0^{\pi/2} \cos \varphi \ln(1 - \cos \varphi) d\varphi \\ &- \frac{1}{2\pi} \int_0^{\pi/2} \ln(1 + \cos \varphi) d\varphi - \frac{1}{2\pi} \int_0^{\pi/2} \ln(1 - \cos \varphi) d\varphi \\ &+ \frac{1}{2\pi} \int_0^{\pi/2} \cos^2 \varphi \ln\left(\frac{1 + \cos \varphi}{1 - \cos \varphi}\right) d\varphi. \end{aligned} \quad (9)$$

Firstly, integrating by parts, we get

$$\int_0^{\pi/2} \cos \varphi \ln(1 - \cos \varphi) d\varphi = -\left(1 + \frac{\pi}{2}\right). \quad (10)$$

In the following calculations we use various integral representations of Catalan's constant, which can be found in [10]. Using the tangent

half-angle substitution $x = \tan(\varphi/2)$ and formula (23) from [10], we obtain

$$\begin{aligned} \int_0^{\pi/2} \ln(1 + \cos \varphi) d\varphi &= \int_0^1 \frac{2}{1+x^2} \ln\left(\frac{2}{1+x^2}\right) dx \\ &= \frac{\pi}{2} \ln 2 - 2 \int_0^1 \frac{\ln(1+x^2)}{1+x^2} dx \\ &= -\frac{\pi}{2} \ln 2 + 2C. \end{aligned} \quad (11)$$

From this equality and formula (10) from [10] we get

$$\int_0^{\pi/2} \ln(1 - \cos \varphi) d\varphi = -\frac{\pi}{2} \ln 2 - 2C. \quad (12)$$

Finally, integrating by parts and using formula (4) from [10], we have

$$\int_0^{\pi/2} \cos^2 \varphi \ln\left(\frac{1 + \cos \varphi}{1 - \cos \varphi}\right) d\varphi = 1 + \int_0^{\pi/2} \frac{\varphi}{\sin \varphi} d\varphi = 1 + 2C. \quad (13)$$

Now, combining (9), (10), (11), (12) and (13), we obtain

$$\langle H_{\text{dyn}}(U) \rangle_{\mathcal{U}(2)} = \frac{3}{2} \ln 2 + \frac{2C - \pi - 1}{2\pi} \approx 0.672. \quad \blacksquare$$

Thus, the average entropy is in this case not far from its maximal value $\ln 2 \approx 0.693$.

V. PVM-DYNAMICAL ENTROPY: QUTRITS AND BEYOND

To determine whether or not a given unitary operator U belongs to $C_d = \{U \in \mathcal{U}(d) : U \text{ is chaotic}\}$, one has to know its spectrum lying on the unit circle and defined up to a phase factor. We can, because of this overall phase freedom, restrict our attention to the set of special unitary matrices and assume that $U \in \text{SU}(d)$. It is well known that all possible values of the trace of matrices from $\text{SU}(d)$ fill in the region $T_d := \{\text{tr } U : U \in \text{SU}(d)\}$ in the complex plane bounded by a d -hypocycloid with cusps at d -th roots of unity scaled up by d , i.e., the curve produced by a point on the circumference of a small circle of radius 1 rolling around the inside of a large circle of radius d and starting at $(d, 0)$ [12, Theorem 5.2], see also [25]. It follows from Corollary 4 that $CT_d := \{\text{tr } U : U \in \text{SU}(d), U \text{ is chaotic}\}$, i.e., the image of the set of special chaotic matrices under the trace map, is contained in the ball $B(0, \sqrt{d})$. We shall see that CT_d is the subset of $T_d \cap B(0, \sqrt{d})$ (which is just $B(0, \sqrt{d})$ for $d \geq 4$) given by the union of regions indexed by pairs consisting of a complex Hadamard matrix of order d and a permutation of a d -element set. Each of these regions is the image of T_d under a spiral similarity with centre at 0, ratio $1/\sqrt{d}$, and angle of rotation that depends on the index. Namely, for a given pair (H, σ) consider the Leibnitz formula for the determinant of H . A d -th root of the normalized summand corresponding to σ is equal to the complex multiplier defining the spiral similarity.

In fact it is enough to take here 'benchmark' Hadamard matrices defined in the following way. Denote by \mathcal{H}_d the set of all complex Hadamard matrices of order d . We call $\mathcal{B} \subset \mathcal{H}_d$ a *benchmark set* if every $H \in \mathcal{H}_d$ is *equivalent* to some matrix F in \mathcal{B} , i.e., it is of the form $H = D_1 P_1 F P_2 D_2$, where D_1, D_2 are diagonal unitary matrices and P_1, P_2 are permutation matrices. We have

Theorem 8. *Let $\mathcal{B} \subset \mathcal{H}_d$ be a benchmark set. Then*

$$\begin{aligned} CT_d &= \bigcup \{ \alpha_{F, \sigma} T_d : F \in \mathcal{B}, \sigma \in S_d \} \\ &= \bigcup \{ \alpha_{F, \sigma} T_d : F \in \mathcal{H}_d, \sigma \in S_d \}, \end{aligned} \quad (14)$$

where for $F \in \mathcal{H}_d, \sigma \in S_d$ we take $\alpha_{F, \sigma}$ to be any d -th root of $(\det F)^{-1} (\text{sgn } \sigma) \prod_{j=1}^d F_{j, \sigma(j)}$ (and so $|\alpha_{F, \sigma}| = 1/\sqrt{d}$).

Proof: Let $U \in \text{SU}(d) \cap C_d$. It follows from Proposition 3 that U is represented in some orthonormal basis by $H \in \text{H}_d$ rescaled by the factor $1/\sqrt{d}$. Fix this basis. Then one can find $F \in \text{B}$, diagonal unitary matrices D_1, D_2 and permutation matrices P_{σ_r} corresponding to $\sigma_r \in S_d$ ($r = 1, 2$) such that $H = D_1 P_1 F P_2 D_2$. Put $D := D_2 D_1$ and $\sigma := \sigma_2 \circ \sigma_1$. Observe that $\text{sgn}(\sigma) = \det(P_2 P_1)$. Moreover, $d^{d/2} = \det H = \det(P_2 P_1) \det D \det F$. Let $\lambda_j \in \mathbb{C}$, $|\lambda_j| = 1$ ($j = 1, \dots, d$) stand for the diagonal elements of D . Set

$$D' := d^{1/2} \overline{\alpha_{F,\sigma}} \text{diag}(\lambda_{\sigma_1(j)} F_{j,\sigma(j)})_{j=1}^d.$$

Then D' is a unitary matrix as $|\alpha_{F,\sigma}| = 1/\sqrt{d}$. We have

$$\begin{aligned} \det D' &= d^{d/2} (\overline{\alpha_{F,\sigma}})^d \prod_{j=1}^d \lambda_{\sigma_1(j)} F_{j,\sigma(j)} \\ &= d^{d/2} (\overline{\alpha_{F,\sigma}})^d (\det D) \prod_{j=1}^d F_{j,\sigma(j)} \\ &= d^{d/2} (\overline{\alpha_{F,\sigma}})^d (\det F)^{-1} \text{sgn}(\sigma) \prod_{j=1}^d F_{j,\sigma(j)} \\ &= d^d |\alpha_{F,\sigma}|^{2d} = 1. \end{aligned}$$

Hence, $\text{tr } D' \in T_d$. Moreover,

$$\begin{aligned} \sqrt{d} \text{tr } U &= \text{tr } H = \text{tr } P_{\sigma_1} F P_{\sigma_2} D = \sum_{j=1}^d \lambda_j F_{\sigma_1^{-1}(j)\sigma_2(j)} \\ &= \sum_{j=1}^d \lambda_{\sigma_1(j)} F_{j,\sigma(j)} = \sqrt{d} \alpha_{F,\sigma} \text{tr } D', \end{aligned}$$

and so $\text{tr } U \in \alpha_{F,\sigma} T_d$. In this way, we showed that $CT_d \subset \bigcup \{\alpha_{F,\sigma} T_d : F \in \text{B}, \sigma \in S_d\}$.

Now let $F \in \text{H}_d$, $\sigma \in S_d$ and $\lambda \in T_d$. Then there is a unitary $U \in \text{SU}(d)$ such that $\text{tr } U = \lambda$. Fix an eigenbasis of U . Then U is represented by a matrix $\text{diag}(\kappa_j)_{j=1}^d$, where $\kappa_j \in \mathbb{C}$, $|\kappa_j| = 1$ ($j = 1, \dots, d$), $\sum_{j=1}^d \kappa_j = \lambda$ and $\prod_{j=1}^d \kappa_j = 1$. Define $D' := \text{diag}(\lambda_j)_{j=1}^d F P_{\sigma}$ with $\lambda_j := \alpha_{F,\sigma} \kappa_j F_{j,\sigma(j)}$ for $j = 1, \dots, d$. Then $\sqrt{d} D' \in \text{H}_d$ fulfills

$$\begin{aligned} \det D' &= (\prod_{j=1}^d \lambda_j) \text{sgn}(\sigma) (\det F) \\ &= \alpha_{F,\sigma}^d \text{sgn}(\sigma) (\det F) \prod_{j=1}^d \overline{F_{j,\sigma(j)}} = 1 \end{aligned}$$

and $\text{tr } D' = \sum_{j=1}^d \lambda_j F_{j,\sigma(j)} = \alpha_{F,\sigma} \lambda$. Thus, D' represents, by Proposition 3, a chaotic $U' \in \text{SU}(d)$ such that $\text{tr } U' = \alpha_{F,\sigma} \lambda$. Hence, $\alpha_{F,\sigma} \lambda \in CT_3$. In consequence, $\bigcup \{\alpha_{F,\sigma} T_d : F \in \text{H}_d, \sigma \in S_d\} \subset CT_d$, which completes the proof. ■

This theorem gives us another characterization of the set of chaotic unitaries for $d = 2$. In this case, since the Fourier matrix F_2 , where

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

serves as the only benchmark Hadamard matrix, we get immediately $CT_2 = \{x \in \mathbb{R} : |x| \leq \sqrt{2}\} = (1/\sqrt{2}) \{x \in \mathbb{R} : |x| \leq 2\} = (1/\sqrt{2}) T_2$. Hence, we obtain the following simple result, which can also be easily deduced from (8).

Proposition 9. *Let $U \in \text{U}(2)$. Then U is chaotic iff $|\text{tr } U| \leq \sqrt{2}$.*

In the case of qutrits ($d = 3$) it follows from Theorem 8 that CT_3 , i.e., the image of the set of special chaotic matrices under the trace map, is the subset of T_3 given by the union of two regions each of which is bounded by a 3-hypocycloid that arises from the original 3-hypocycloid (the black curve in Fig. 3) by scaling it down by a factor of $\sqrt{3}$ and rotating by $\pm\pi/18$ (the union of figures bounded by the red curves in Fig. 3).

Observe that the characteristic polynomial of $U \in \text{SU}(3)$ takes the form $\lambda^3 - (\text{tr } U) \lambda^2 + \overline{(\text{tr } U)} \lambda - 1$, so the spectrum of U , and thus the answer to the question whether it is chaotic or not, depends solely on its trace. Thus, it is not a surprise that in this case the necessary condition (14) becomes sufficient as well.

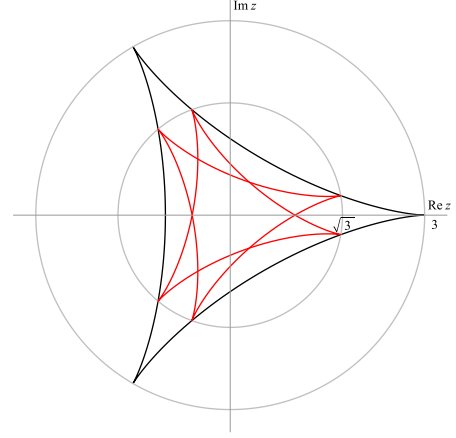


Fig. 3. Traces of special chaotic unitaries for $d = 3$ (the region bounded by the red curves).

Theorem 10. *Let $U \in \text{U}(3)$ and let β be a cube root of $\det U$. Then U is chaotic iff*

$$\frac{1}{\beta} \text{tr } U \in CT_3 = \frac{1}{\sqrt{3}} (\alpha T_3 \cup \overline{\alpha} T_3),$$

where $\alpha := e^{\frac{\pi}{18}i}$.

Proof: All complex Hadamard matrices of order 3 are equivalent to the Fourier matrix F_3 [14], where

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3 & \omega_3^2 \\ 1 & \omega_3^2 & \omega_3 \end{bmatrix}$$

and $\omega_3 := \exp(2\pi i/3)$. Applying Theorem 8 and $\det F_3 = -3\sqrt{3}i$, we obtain two possible scaling factors: $\alpha_{F,\text{id}}^3 = -\omega_3^2/(3\sqrt{3}i) = (\overline{\alpha}/\sqrt{3})^3$ and $\alpha_{F,\sigma}^3 = \omega_3/(3\sqrt{3}i) = (\alpha/\sqrt{3})^3$, with $\sigma \in S_3$ defined by $\sigma(1) = 1$, $\sigma(2) = 3$ and $\sigma(3) = 2$, which implies $CT_3 = \frac{1}{\sqrt{3}} (\alpha T_3 \cup \overline{\alpha} T_3)$. Now, the assertion follows from the fact that the spectrum of $U \in \text{SU}(3)$ is fully defined by its trace. ■

Next, we use the above result to estimate the volume of the set of chaotic unitaries in dimension 3. First, observe that

$$m(C_3) = \mu(U \in \text{SU}(3), U \text{ is chaotic}),$$

where μ stands for the normalized Haar measure on $\text{SU}(3)$. Now, from the Weyl integration formula for $\text{SU}(3)$ [25, eq.(9)] and Theorem 10, we obtain

Theorem 11.

$$m(C_3) = \frac{3\sqrt{3}}{2\pi^2} \int_{CT_3} \sqrt{4 + \left(\frac{2r}{3}\right)^3 \cos 3\theta - 3 \left(1 + \frac{r^2}{9}\right)^2} r dr d\theta.$$

Evaluating the above integral numerically, we get $m(C_3) \approx 0.592$. It is noteworthy that $m(C_3) < m(C_2)$.

Observe that Theorem 8 does not provide, however, any new information about chaotic unitaries for $d = 4$, since the one-parameter family of benchmark Hadamard matrices

$$F_4^{(1)}(\varphi) := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & ie^{i\varphi} & -1 & -ie^{i\varphi} \\ 1 & -1 & 1 & -1 \\ 1 & -ie^{i\varphi} & -1 & ie^{i\varphi} \end{bmatrix},$$

where $\varphi \in [0, 2\pi)$, see [45], generates all possible complex multipliers of modulus $1/2$.

On the other hand, for $d = 5$ the benchmark set consists only of the Fourier matrix

$$F_5 := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_5 & \omega_5^2 & \omega_5^3 & \omega_5^4 \\ 1 & \omega_5^2 & \omega_5^4 & \omega_5 & \omega_5^3 \\ 1 & \omega_5^3 & \omega_5 & \omega_5^4 & \omega_5^2 \\ 1 & \omega_5^4 & \omega_5^3 & \omega_5^2 & \omega_5 \end{bmatrix},$$

where $\omega_5 := \exp(2\pi i/5)$, see [45]. By direct calculation we deduce from Theorem 8 a simple necessary condition for $U \in \mathcal{U}(5)$ to be chaotic.

Proposition 12. *Let $U \in \mathcal{U}(5)$ and $\beta^5 = \det U$. If U is chaotic, then*

$$\frac{1}{\beta} \operatorname{tr} U \in CT_5 = \frac{1}{\sqrt{5}} \bigcup \{\alpha T_5 : \alpha \in A\},$$

where $A := \{1, -1, e^{\frac{\pi}{25}i}, e^{-\frac{\pi}{25}i}, e^{\frac{2\pi}{25}i}, e^{-\frac{2\pi}{25}i}\}$, see Fig. 4.

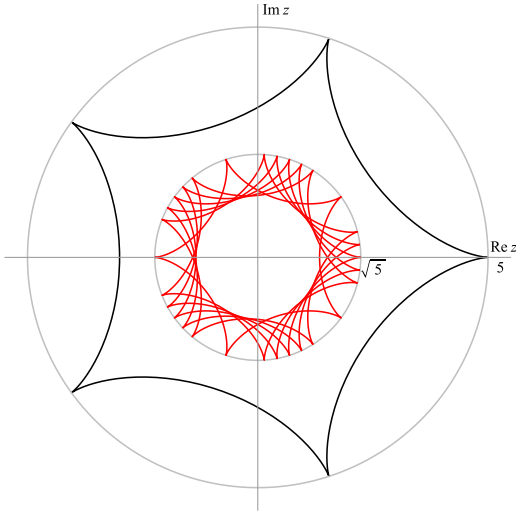


Fig. 4. Traces of special chaotic unitaries for $d = 5$ (the region bounded by the red curves).

For higher dimensions ($d \geq 6$) Theorem 8 does not provide concrete information about the chaoticity of a unitary map, since the complete classification of complex Hadamard matrices is only available up to order $d = 5$.

VI. ENTROPY OF MEASUREMENT AND POVM-ENTROPY

In the closing section we would like to briefly discuss some issues related to the POVM-dynamical entropy. We start by recalling the notion of entropy of a POVM. By the (Shannon) entropy of the measurement $\Pi = (\Pi_j)_{j=1,\dots,k}$, where $\Pi_j = (d/k) |\varphi_j\rangle\langle\varphi_j|$ for $|\varphi_j\rangle\langle\varphi_j| \in \mathcal{P}(\mathbb{C}^d)$ ($j = 1, \dots, k$), we mean the function $H(\cdot, \Pi) : \mathcal{S}(\mathbb{C}^d) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} H(\rho, \Pi) &:= \sum_{j=1}^k \eta(p_j(\rho, \Pi)) \\ &= \sum_{j=1}^k \eta((d/k) \langle\varphi_j|\rho|\varphi_j\rangle) \\ &= \ln \frac{k}{d} + \frac{d}{k} \sum_{j=1}^k \eta(\langle\varphi_j|\rho|\varphi_j\rangle) \end{aligned}$$

for an input state $\rho \in \mathcal{S}(\mathbb{C}^d)$; see [38], [49] for the history and information-theoretic interpretation of this notion. If $\rho = |\psi\rangle\langle\psi| \in \mathcal{P}(\mathbb{C}^d)$, we put $H(|\psi\rangle, \Pi) := H(\rho, \Pi)$. Applying (1), we see that

the entropy of $U \in \mathcal{U}(d)$ with respect to Π can be expressed as the mean entropy of Π averaged over the output states of Π transformed by U :

$$H(U, \Pi) = \frac{1}{k} \sum_{j=1}^k H(U|\varphi_j\rangle, \Pi) \quad (15)$$

and so

$$H_{\text{meas}}(\Pi) = H(\mathbb{I}, \Pi) = \frac{1}{k} \sum_{j=1}^k H(|\varphi_j\rangle, \Pi). \quad (16)$$

From (15) and (16) we obtain

$$H_{\text{dyn}}(U, \Pi) = \frac{1}{k} \sum_{j=1}^k [H(U|\varphi_j\rangle, \Pi) - H(|\varphi_j\rangle, \Pi)]. \quad (17)$$

For PVMs we have $H(\mathbb{I}, \Pi) = 0 \leq H(U, \Pi) = H_{\text{dyn}}(U, \Pi)$. Surprisingly, in the general case we can find situations where intertwining a POVM-measurement with some (or even any) unitary operator can produce smaller entropy than that generated by the measurement itself.

To illustrate this phenomenon, assume that $\Pi = (\Pi_j)_{j=1,\dots,d^2}$ is a SIC-POVM, i.e., a rank-1 POVM satisfying the condition $\operatorname{tr}(\Pi_j \Pi_l) = 1/(d^2(d+1))$ for $j, l = 1, \dots, d^2$, $j \neq l$. Then, from (17) and [43], we get

$$H(U, \Pi) \leq H(\mathbb{I}, \Pi) = \frac{d-1}{d} \ln(d+1) + \ln d.$$

We also have [18], [44] the following bound

$$\ln \frac{(d+1)d}{2} \leq H(U, \Pi),$$

which is known to be actually attained for the ‘tetrahedral’ SIC-POVM in dimension 2 [38], for all SIC-POVMs in dimension 3 [42], and for the Hoggar SIC-POVM in dimension 8 [44]. Consequently, for every $U \in \mathcal{U}(d)$ we get

$$-\ln 2 + \frac{\ln(d+1)}{d} \leq H_{\text{dyn}}(U, \Pi) \leq 0. \quad (18)$$

Thus, from this point of view, SIC-POVMs and PVMs lie on the opposite ends of the spectrum. It seems that the interplay between the two kinds of randomness, one coming from the measurement and one associated with unitary evolution, makes the study of the POVM-dynamical entropy particularly difficult.

VII. CONCLUSIONS

In the present paper we solve some problems concerning chaotic unitaries and PVM-dynamical entropy; however, many questions remain unanswered. We get several sufficient and/or necessary conditions for a matrix to maximize the PVM-dynamical entropy (Proposition 3, Corollary 4, Theorem 8), but only for qubits (Propositions 5 and 9) and qutrits (Theorem 10) we can fully describe the set of chaotic unitaries. The problem of characterising this property in higher dimensions, starting from ququads, remains open. The fact that the probability of finding a chaotic matrix among unitaries is smaller for qutrits (Theorem 11) than for qubits (Theorem 6) suggests the conjecture that this probability decreases, possibly to zero, when the dimension of the Hilbert space grows to infinity. This contrasts with the result of Theorem 2 that the mean PVM-dynamical entropy increases logarithmically with the dimension and is, in fact, almost as large as possible.

On the other hand, extending the definition of the dynamical entropy to other classes of measurements opens up a number of natural questions for further analysis. Moving on to a broader class of POVMs, we are faced, especially in the case of SIC-POVMs (eq. (18)), with the paradoxical fact that a unitary dynamics combined

with a measurement can decrease the randomness, which provides an example of a phenomenon with no classical counterpart. It is also not clear whether the inequality between the PVM-dynamical entropy and the POVM-dynamical entropy can be sharp.

Leaving the realm of rank-1 operators, we encounter an even more interesting situation, first described by one of us (W.S.) in the more general setting of operational approach to (quantum) dynamics and measurement process [37], and then by Wiesner and co-authors in a series of papers [16], [15], [17], [47], [31]. In this case the measurement process together with the unitary dynamics still produces a Markov chain in the space of states, but the accompanying process generated in the space of the measurement outcomes does not have to be Markovian [6], which makes computing the dynamical entropy more challenging.

Finally, a natural direction for further research is to study the semiclassical limit of the dynamical entropies defined here, see also [40].

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